

# BOOLEAN MODELS AND SIMULTANEOUS INEQUALITIES

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ABSTRACT. Boolean valued analysis is applied to deriving operator versions of the classical Farkas Lemma in the theory of simultaneous linear inequalities.

## 1. INTRODUCTION

The Farkas Lemma, also known as the Farkas–Minkowski Lemma, plays a key role in linear programming and the relevant areas of optimization.<sup>1</sup> The aim of this talk is to demonstrate how Boolean valued analysis<sup>2</sup> may be applied to simultaneous linear inequalities with operators. This particular theme is another illustration of the deep and powerful technique of “stratified validity” which is characteristic of Boolean valued analysis.

Assume that  $X$  is a real vector space,  $Y$  is a *Kantorovich space* also known as a Dedekind complete vector lattice or a complete Riesz space. Let  $\mathbb{B} := \mathbb{B}(Y)$  be the *base* of  $Y$ , i.e., the complete Boolean algebras of positive projections in  $Y$ ; and let  $m(Y)$  be the universal completion of  $Y$ . Denote by  $L(X, Y)$  the space of linear operators from  $X$  to  $Y$  and let  $\text{Orth}(Y)$  stand for the commutant of  $\mathbb{B}$  in  $L^{(r)}(Y)$ . In case  $X$  is furnished with some  $Y$ -seminorm on  $X$ , by  $L^{(m)}(X, Y)$  we mean the *space of dominated operators* from  $X$  to  $Y$ . As usual,  $\{T \leq y\} := \{x \in X \mid Tx \leq y\}$  and  $\ker(T) = T^{-1}(0)$  for  $T : X \rightarrow Y$ .

## 2. INEQUALITIES: EXPLICIT DOMINANCE

$$\begin{array}{ccc} X & \xrightarrow{A} & W \\ & \searrow B & \vdots \mathfrak{X} \\ & & Y \end{array}$$

- (1):  $(\exists \mathfrak{X}) \mathfrak{X}A = B \leftrightarrow \ker(A) \subset \ker(B)$ ;  
 (2):<sup>3</sup> If  $W$  is ordered by  $W_+$  and  $A(X) - W_+ = W_+ - A(X) = W$ , then

$$(\exists \mathfrak{X} \geq 0) \mathfrak{X}A = B \leftrightarrow \{A \leq 0\} \subset \{B \leq 0\}.$$

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<sup>1</sup>Cp. [1][2].

<sup>2</sup>Cp. [3].

<sup>3</sup>The Kantorovich Theorem, [4, p. 44].

## 3. FARKAS: EXPLICIT DOMINANCE

THEOREM 1. Assume that  $A_1, \dots, A_N$  and  $B$  belong to  $L^{(m)}(X, Y)$ .

The following are equivalent:

- (1) Given  $b \in \mathbb{B}$ , the operator inequality  $bBx \leq 0$  is a consequence of the simultaneous linear operator inequalities  $bA_1x \leq 0, \dots, bA_Nx \leq 0$ , i.e.,

$$\{bB \leq 0\} \supset \{bA_1 \leq 0\} \cap \dots \cap \{bA_N \leq 0\}.$$

- (2) There are positive orthomorphisms  $\alpha_1, \dots, \alpha_N \in \text{Orth}(m(Y))$  such that

$$B = \sum_{k=1}^N \alpha_k A_k;$$

i.e.,  $B$  lies in the operator convex conic hull of  $A_1, \dots, A_N$ .

## 4. FARKAS: HIDDEN DOMINANCE

LEMMA 1. Let  $X$  be a vector space over some subfield  $R$  of the reals  $\mathbb{R}$ . Assume that  $f$  and  $g$  are  $R$ -linear functionals on  $X$ ; in symbols,  $f, g \in X^\# := L(X, \mathbb{R})$ .

For the inclusion

$$\{g \leq 0\} \supset \{f \leq 0\}$$

to hold it is necessary and sufficient that there be  $\alpha \in \mathbb{R}_+$  satisfying  $g = \alpha f$ .

PROOF. SUFFICIENCY is obvious.

NECESSITY: The case of  $f = 0$  is trivial. If  $f \neq 0$  then there is some  $x \in X$  such that  $f(x) \in \mathbb{R}$  and  $f(x) > 0$ . Denote the image  $f(X)$  of  $X$  under  $f$  by  $R_0$ . Put  $h := g \circ f^{-1}$ , i.e.  $h \in R_0^\#$  is the only solution for  $h \circ f = g$ . By hypothesis,  $h$  is a positive  $R$ -linear functional on  $R_0$ . By the Bigard Theorem [4, p. 108]  $h$  can be extended to a positive homomorphism  $\bar{h} : \mathbb{R} \rightarrow \mathbb{R}$ , since  $R_0 - \mathbb{R}_+ = \mathbb{R}_+ - R_0 = \mathbb{R}$ . Each positive automorphism of  $\mathbb{R}$  is multiplication by a positive real. As the sought  $\alpha$  we may take  $\bar{h}(1)$ .

The proof of the lemma is complete.

## 5. REALS: EXPLICIT DOMINANCE

LEMMA 2. Let  $X$  be an  $\mathbb{R}$ -seminormed vector space over some subfield  $R$  of  $\mathbb{R}$ . Assume that  $f_1, \dots, f_N$  and  $g$  are bounded  $R$ -linear functionals on  $X$ ; in symbols,  $f_1, \dots, f_N, g \in X^* := L^{(m)}(X, \mathbb{R})$ .

For the inclusion

$$\{g \leq 0\} \supset \bigcap_{k=1}^N \{f_k \leq 0\}$$

to hold it is necessary and sufficient that there be  $\alpha_1, \dots, \alpha_N \in \mathbb{R}_+$  satisfying

$$g = \sum_{k=1}^N \alpha_k f_k.$$

## 6. ORIGINS

Cohen's final solution of the problem of the cardinality of the continuum within ZFC gave rise to the Boolean valued models.

Scott forecasted in 1969:<sup>4</sup>

*We must ask whether there is any interest in these nonstandard models aside from the independence proof; that is, do they have any mathematical interest? The answer must be yes, but we cannot yet give a really good argument.*

Takeuti coined the term "Boolean valued analysis" for applications of the models to analysis.<sup>5</sup>

## 7. BOOLEAN VALUED UNIVERSE

Let  $\mathbb{B}$  be a complete Boolean algebra. Given an ordinal  $\alpha$ , put

$$V_\alpha^{(\mathbb{B})} := \{x \mid (\exists \beta \in \alpha) x : \text{dom}(x) \rightarrow \mathbb{B} \ \& \ \text{dom}(x) \subset V_\beta^{(\mathbb{B})}\}.$$

The *Boolean valued universe*  $\mathbb{V}^{(\mathbb{B})}$  is

$$\mathbb{V}^{(\mathbb{B})} := \bigcup_{\alpha \in \text{On}} V_\alpha^{(\mathbb{B})},$$

with On the class of all ordinals. The truth value  $\llbracket \varphi \rrbracket \in \mathbb{B}$  is assigned to each formula  $\varphi$  of ZFC relativized to  $\mathbb{V}^{(\mathbb{B})}$ .

## 8. DESCENDING AND ASCENDING

Given  $\varphi$ , a formula of ZFC, and  $y$ , a member of  $\mathbb{V}^{(\mathbb{B})}$ ; put  $A_\varphi := A_{\varphi(\cdot, y)} := \{x \mid \varphi(x, y)\}$ . The *descent*  $A_\varphi \downarrow$  of a class  $A_\varphi$  is

$$A_\varphi \downarrow := \{t \mid t \in \mathbb{V}^{(\mathbb{B})} \ \& \ \llbracket \varphi(t, y) \rrbracket = 1\}.$$

If  $t \in A_\varphi \downarrow$ , then it is said that  $t$  *satisfies*  $\varphi(\cdot, y)$  *inside*  $\mathbb{V}^{(\mathbb{B})}$ . The *descent*  $x \downarrow$  of  $x \in \mathbb{V}^{(\mathbb{B})}$  is defined as

$$x \downarrow := \{t \mid t \in \mathbb{V}^{(\mathbb{B})} \ \& \ \llbracket t \in x \rrbracket = 1\},$$

i.e.  $x \downarrow = A_{\in x} \downarrow$ . The class  $x \downarrow$  is a set. If  $x$  is a nonempty set inside  $\mathbb{V}^{(\mathbb{B})}$  then

$$(\exists z \in x \downarrow) \llbracket (\exists t \in x) \varphi(t) \rrbracket = \llbracket \varphi(z) \rrbracket.$$

The *ascent* functor acts in the opposite direction.

## 9. THE REALS WITHIN

There is an object  $\mathcal{R}$  inside  $\mathbb{V}^{(\mathbb{B})}$  modeling  $\mathbb{R}$ , i.e.,

$$\llbracket \mathcal{R} \text{ is the reals} \rrbracket = 1.$$

Let  $\mathcal{R} \downarrow$  be the descent of the carrier  $|\mathcal{R}|$  of the algebraic system

$$\mathcal{R} := (|\mathcal{R}|, +, \cdot, 0, 1, \leq)$$

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<sup>4</sup>Cp. [5].

<sup>5</sup>Cp. [6].

inside  $\mathbb{V}^{(\mathbb{B})}$ . Implement the descent of the structures on  $|\mathcal{R}|$  to  $\mathcal{R}\downarrow$  as follows:

$$\begin{aligned} x + y = z &\leftrightarrow \llbracket x + y = z \rrbracket = 1; \\ xy = z &\leftrightarrow \llbracket xy = z \rrbracket = 1; \\ x \leq y &\leftrightarrow \llbracket x \leq y \rrbracket = 1; \\ \lambda x = y &\leftrightarrow \llbracket \lambda^\wedge x = y \rrbracket = 1 \\ (x, y, z \in \mathcal{R}\downarrow, \lambda \in \mathbb{R}). \end{aligned}$$

**GORDON THEOREM.**  $\mathcal{R}\downarrow$  with the descended structures is a universally complete vector lattice with base  $\mathbb{B}(\mathcal{R}\downarrow)$  isomorphic to  $\mathbb{B}$ .

Proof of Theorem 1.

(2)  $\rightarrow$  (1): If  $B = \sum_{k=1}^N \alpha_k A_k$  for some positive  $\alpha_1, \dots, \alpha_N$  in  $\text{Orth}(m(Y))$  while  $bA_k x \leq 0$  for  $b \in \mathbb{B}$  and  $x \in X$ , then

$$bBx = b \sum_{k=1}^N \alpha_k A_k x = \sum_{k=1}^N \alpha_k bA_k x \leq 0$$

since orthomorphisms commute and projections are orthomorphisms of  $m(Y)$ .

(1)  $\rightarrow$  (2): Consider the separated Boolean valued universe  $\mathbb{V}^{(\mathbb{B})}$  over the base  $\mathbb{B}$  of  $Y$ . By the Gordon Theorem the ascent  $Y^\uparrow$  of  $Y$  is  $\mathcal{R}$ , the reals inside  $\mathbb{V}^{(\mathbb{B})}$ .

Using the canonical embedding, we see that  $X^\wedge$  is an  $\mathcal{R}$ -seminormed vector space over the standard name  $\mathbb{R}^\wedge$  of the reals  $\mathbb{R}$ . Moreover,  $\mathbb{R}^\wedge$  is a subfield and sublattice of  $\mathcal{R} = Y^\uparrow$  inside  $\mathbb{V}^{(\mathbb{B})}$ .

Put  $f_k := A_k^\uparrow$  for all  $k := 1, \dots, N$  and  $g := B^\uparrow$ . Clearly, all  $f_1, \dots, f_N, g$  belong to  $(X^\wedge)^*$  inside  $\mathbb{V}^{(\mathbb{B})}$ .

Define the finite sequence

$$f : \{1, \dots, N\}^\wedge \rightarrow (X^\wedge)^*$$

as the ascent of  $(f_1, \dots, f_N)$ . In other words, the truth values are as follows:

$$\llbracket f_{k^\wedge}(x^\wedge) = A_k x \rrbracket = 1, \quad \llbracket g(x^\wedge) = Bx \rrbracket = 1$$

for all  $x \in X$  and  $k := 1, \dots, N$ .

Put

$$b := \llbracket A_1 x \leq 0^\wedge \rrbracket \wedge \dots \wedge \llbracket A_N x \leq 0^\wedge \rrbracket.$$

Then  $bA_k x \leq 0$  for all  $k := 1, \dots, N$  and  $bBx \leq 0$  by (1).

Therefore,

$$\llbracket A_1 x \leq 0^\wedge \rrbracket \wedge \dots \wedge \llbracket A_N x \leq 0^\wedge \rrbracket \leq \llbracket Bx \leq 0^\wedge \rrbracket.$$

In other words,

$$\begin{aligned} &\llbracket (\forall k := 1^\wedge, \dots, N^\wedge) f_k(x^\wedge) \leq 0^\wedge \rrbracket \\ &= \bigwedge_{k:=1, \dots, N} \llbracket f_{k^\wedge}(x^\wedge) \leq 0^\wedge \rrbracket \leq \llbracket g(x^\wedge) \leq 0^\wedge \rrbracket. \end{aligned}$$

Using Lemma 2 inside  $\mathbb{V}^{(\mathbb{B})}$  and appealing to the maximum principle of Boolean valued analysis, we infer that there is a finite sequence  $\alpha : \{1^\wedge, \dots, N^\wedge\} \rightarrow \mathcal{R}_+$  inside  $\mathbb{V}^{(\mathbb{B})}$  satisfying

$$\llbracket (\forall x \in X^\wedge) g(x) = \sum_{k=1^\wedge}^{N^\wedge} \alpha(k) f_k(x) \rrbracket = 1.$$

Put  $\alpha_k := \alpha(k^\wedge) \in \mathcal{R}_+ \downarrow$  for  $k := 1, \dots, N$ . Multiplication by an element in  $\mathcal{R} \downarrow$  is an orthomorphism of  $m(Y)$ . Moreover,

$$B = \sum_{k=1}^N \alpha_k A_k,$$

which completes the proof.

#### 10. COUNTEREXAMPLE: NO DOMINANCE

Lemma 1, describing the consequences of a single inequality, does not restrict the class of functionals under consideration. The analogous version of the Farkas Lemma simply fails for two simultaneous inequalities in general. Indeed, the inclusion  $\{f = 0\} \subset \{g \leq 0\}$  equivalent to the inclusion  $\{f = 0\} \subset \{g = 0\}$  does not imply that  $f$  and  $g$  are proportional in the case of an arbitrary subfield of  $\mathbb{R}$ . It suffices to look at  $\mathbb{R}$  over the rationals  $\mathbb{Q}$ , take some discontinuous  $\mathbb{Q}$ -linear functional on  $\mathbb{Q}$  and the identity automorphism of  $\mathbb{Q}$ . This gives grounds for the next result.

#### 11. RECONSTRUCTION: NO DOMINANCE

**THEOREM 2.** *Take  $A$  and  $B$  in  $L(X, Y)$ . The following are equivalent:*

- (1)  $(\exists \alpha \in \text{Orth}(m(Y))) B = \alpha A$ ;
- (2) *There is a projection  $\varkappa \in \mathbb{B}$  such that*

$$\{\varkappa b B \leq 0\} \supset \{\varkappa b A \leq 0\}; \quad \{\neg \varkappa b B \leq 0\} \supset \{\neg \varkappa b A \geq 0\}$$

for all  $b \in \mathbb{B}$ .

**PROOF.** Boolean valued analysis reduces the claim to the scalar case. Applying Lemma 1 twice and writing down the truth values, complete the proof.

#### 12. INTERVAL OPERATORS

Let  $X$  be a vector lattice. An *interval operator*  $\mathbf{T}$  from  $X$  to  $Y$  is an order interval  $[\underline{T}, \overline{T}]$  in  $L^{(r)}(X, Y)$ , with  $\underline{T} \leq \overline{T}$ .

The interval equation  $\mathbf{B} = \mathfrak{X}\mathbf{A}$  has a *weak interval solution*<sup>6</sup> provided that  $(\exists \mathfrak{X})(\exists A \in \mathbf{A})(\exists B \in \mathbf{B}) B = \mathfrak{X}A$ .

Given an interval operator  $\mathbf{T}$  and  $x \in X$ , put

$$P_{\mathbf{T}}(x) = \overline{T}x_+ - \underline{T}x_-.$$

Call  $\mathbf{T}$  is *adapted* in case  $\overline{T} - \underline{T}$  is the sum of finitely many disjoint addends, and put  $\sim(x) := -x$  for all  $x \in X$ .

#### 13. INTERVAL EQUATIONS

**THEOREM 3.** *Let  $X$  be a vector lattice, and let  $Y$  be a Kantorovich space. Assume that  $\mathbf{A}_1, \dots, \mathbf{A}_N$  are adapted interval operators and  $\mathbf{B}$  is an arbitrary interval operator in the space of order bounded operators  $L^{(r)}(X, Y)$ .*

*The following are equivalent:*

- (1) *The interval equation*

$$\mathbf{B} = \sum_{k=1}^N \alpha_k \mathbf{A}_k$$

*has a weak interval solution  $\alpha_1, \dots, \alpha_N \in \text{Orth}(Y)_+$ .*

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<sup>6</sup>Cp. [7].

(2) For all  $b \in \mathbb{B}$  we have

$$\{b\mathfrak{B} \geq 0\} \supset \{b\mathfrak{A}_1^\sim \leq 0\} \cap \cdots \cap \{b\mathfrak{A}_N^\sim \leq 0\},$$

where  $\mathfrak{A}_k^\sim := P_{\mathbf{A}_k} \circ \sim$  for  $k := 1, \dots, N$  and  $\mathfrak{B} := P_{\mathbf{B}}$ .

#### 14. INHOMOGENEOUS INEQUALITIES

**THEOREM 4.** *Let  $X$  be a  $Y$ -seminormed space, with  $Y$  a Kantorovich space. Assume given some dominated operators  $A_1, \dots, A_N, B \in L^{(m)}(X, Y)$  and elements  $u_1, \dots, u_N, v \in Y$ . The following are equivalent:*

(1) For all  $b \in \mathbb{B}$  the inhomogeneous operator inequality  $bBx \leq bv$  is a consequence of the consistent simultaneous inhomogeneous operator inequalities  $bA_1x \leq bu_1, \dots, bA_Nx \leq bu_N$ , i.e.,

$$\{bB \leq bv\} \supset \{bA_1 \leq bu_1\} \cap \cdots \cap \{bA_N \leq bu_N\}.$$

(2) There are positive orthomorphisms  $\alpha_1, \dots, \alpha_N \in \text{Orth}(m(Y))$  satisfying

$$B = \sum_{k=1}^N \alpha_k A_k; \quad v \geq \sum_{k=1}^N \alpha_k u_k.$$

#### 15. INHOMOGENEOUS MATRIX INEQUALITIES

In applications we encounter inhomogeneous matrix inequalities over various finite-dimensional spaces.<sup>7</sup>

**THEOREM 5.** *Let  $X$  be a real  $Y$ -seminormed space, with  $Y$  a Kantorovich space. Assume that  $A \in L^{(m)}(X, Y^s)$ ,  $A \in L^{(m)}(X, Y^s)$ ,  $u \in Y^t$  and  $v \in Y^m$ , where  $s$  and  $t$  are some naturals.*

*The following are equivalent:*

(1) For all  $b \in \mathbb{B}$  the inhomogeneous operator inequality  $bBx \leq bv$  is a consequence of the consistent inhomogeneous inequality  $bAx \leq bu$ , i.e.,  $\{bB \leq bv\} \supset \{bA \leq bu\}$ .

(2) There is some  $s \times t$  matrix with entries positive orthomorphisms of  $m(Y)$  such that  $B = \mathfrak{X}A$  and  $\mathfrak{X}u \leq v$  for the corresponding linear operator  $\mathfrak{X} \in L_+(Y^s, Y^t)$ .

#### 16. COMPLEX SCALARS

**THEOREM 6.** *Let  $X$  be a complex  $Y$ -seminormed space, with  $Y$  a Kantorovich space. Assume given  $u_1, \dots, u_N, v \in Y$  and dominated operators  $A_1, \dots, A_N, B \in L^{(m)}(X, Y_{\mathbb{C}})$  from  $X$  into the complexification  $Y_{\mathbb{C}} := Y \otimes iY$  of  $Y$ . The following are equivalent:*

(1) For all  $b \in \mathbb{B}$  and  $x \in X$  the inhomogeneous inequality  $b|Bx| \leq bv$  is a consequence of the consistent simultaneous inhomogeneous inequalities  $b|A_1x| \leq bu_1, \dots, b|A_Nx| \leq bu_N$ , i.e.,

$$\{b|B| \leq bv\} \supset \{b|A_1| \leq bu_1\} \cap \cdots \cap \{b|A_N| \leq bu_N\}.$$

(2) There are complex orthomorphisms  $c_1, \dots, c_N \in \text{Orth}(m(Y)_{\mathbb{C}})$  satisfying

$$B = \sum_{k=1}^N c_k A_k; \quad v \geq \sum_{k=1}^N |c_k| u_k.$$

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<sup>7</sup>Cp. [8].

# 17. THEOREM OF THE ALTERNATIVE

THEOREM 7. *Let  $X$  be a  $Y$ -seminormed real vector space, with  $Y$  a Kantorovich space. Assume that  $A_1, \dots, A_N$  and  $B$  belong to  $L^{(m)}(X, Y)$ .*

*Then one and only one of the following holds:*

- (1) *There are  $x \in X$  and  $b, b' \in \mathbb{B}$  such that  $b' \leq b$  and*

$$b'Bx > 0, bA_1x \leq 0, \dots, bA_Nx \leq 0.$$

- (2) *There are  $\alpha_1, \dots, \alpha_N \in \text{Orth}(m(Y))_+$  such that  $B = \sum_{k=1}^N \alpha_k A_k$ .*

# 18. ALL IS NUMBER

The above results, although curious to some extent, are nothing more than simple illustrations of the powerful technique of model theory shedding new light at the *Pythagorean Thesis*. The theory of the reals enriches mathematics, demonstrating the liberating role of logic.

# 19. PURSUIT OF TRUTH

We definitely feel truth, but we cannot define truth properly. That is what Tarski explained to us in the 1930s.

We pursue truth by way of proof, as wittily phrased by Mac Lane. Model theory evaluates and counts truth and proof.

The chase of truth not only leads us close to the truth we pursue but also enables us to nearly catch up with many other instances of truth which we were not aware nor even foresaw at the start of the rally pursuit. That is what we have learned from the Boolean models elaborated in the 1960s by Scott, Solovay, and Vopěnka.

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